Then

(i) $g \neq 0 \iff \varphi$: isom

(ii) Π_n^{ab} : free $\mathbb{Z}_\ell$-module

(iii) $(g.r) = (1,1)$

$\iff \mathrm{rk}_{\mathbb{Z}_\ell}(\Pi_n^{ab}) = 2n$

[cf. (ii)]

(i)

$\Longrightarrow$ lem 5 + some argument

$\Longleftarrow$ theory of the wt filtration of Π_n
[cf. Nakamura, Takao, Ueno]

(ii) $g \neq 0 \rightsquigarrow$ follows from (i)

$g = 0 \rightsquigarrow$ follows from the theory of wt filt'n

(iii) follows from (i). //

Pf

$(g^\circ . r^\circ$

$(g^\circ . r^\circ) \notin$

$\Longrightarrow$

prop 4

...me argument

...f the wt filtration

...ra, Takao, Ueno] of Π_n

...s from (i)

...ows from

... theory of wt filt'n

...m (i'). //

<u>pf of $(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$</u>

$\boxed{(g^\circ, r^\circ) = (1,1)} \overset{\text{prop 7 (iii)}}{\Longrightarrow} (g^\bullet, r^\bullet) = (1,1)$

$\boxed{(g^\circ, r^\circ) \notin \{(0,3),(1,1)\}}$

$\underset{\text{prop 4}}{\Longrightarrow} \quad \alpha : \Pi_n^\circ \xrightarrow{\sim} \Pi_n^\bullet$: PF-adm

$\Longrightarrow \quad \alpha$ induces $\alpha_2 : \Pi_2^\circ \xrightarrow{\sim} \Pi_2^\bullet$: PF-adm

$\underset{\text{prop 6}}{\Longrightarrow} (g^\circ, r^\circ) = (g^\bullet, r^\bullet)$

$\boxed{(g^\circ, r^\circ) = (0,3)}$

$\underset{\text{prop 9. (iii)}}{\Longrightarrow} (g^\bullet, r^\bullet) \neq (1,1)$

Suppose that $(g^\bullet, r^\bullet) \neq (0,3)$

$\underset{\text{Props 4.6}}{\Longrightarrow} (g^\bullet, r^\bullet) = (g^\circ, r^\circ) = (0,3)$ X

$\therefore (g^\bullet, r^\bullet) = (0,3)$ //

$\boxed{\S 4}$ In this §4,

we assume that $\underset{\sim\sim\sim}{(g.r) = (0.3)}$

for simplicity

Note: In this case,

we have a natural k-isom

$X_n \xrightarrow{\sim} M_{0,n+3}$ (the moduli stack of hyperbolic curves of type $(0, n+3)/k$, marked pts are ordered

Write $\Pi_n := \pi_1(X_n)$ or $\pi_1(X_n)^{(\ell)}$

Def (i) We shall refer to

$\ker(\underset{\sim\sim\sim}{\Pi_n \twoheadrightarrow \Pi_{n'}})$

induced by a morphism $M_{0,n+3} \to M_{0,n'+3}$ obtained by forgetting $\overbrace{\text{marked pts}}^{n-n'}$

as a

(ii) Ow

$= \pi_1(X_n)$ or $\pi_1(X_n)^{(\ell)}$

We shall refer to

$\mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_{n'})$

induced by a morphism $M_{0,n+3} \to M_{0,n'+3}$ obtained by forgetting $n-n'$ marked pts

as a generalized fiber subgp

of co-length n'

(or length $n-n'$)

(ii) $\mathrm{Out}^{GF}(\Pi_n) := \{ \alpha \in \mathrm{Out}(\Pi_n) \mid \alpha(K) = K$

$\forall K$: gen fib subgp $\subseteq \Pi_n \}$

$\boxed{\text{Thm B}}$ Let $\sigma \in \mathrm{Aut}(\Pi_n)$

Then σ induces a permutation of the set of gen fiber subgps of Π_n

☹ follows from

[NT], Thm D + Some argument

Nakamura, Takao

Key pt In the case $(0,3), (1,1)$, the weight filtration of Π_n is automatically preserved! by σ

Def G : prof gp

$$Z^{loc}(G) := \varinjlim_{\substack{H \subseteq G \\ \text{open}}} \underline{Z_G(H)}$$

centralizer

(the local center of G)

$\boxed{\text{Thm C}}$ $n \geq 2$

sym gp

(i) $\mathrm{Out}(\Pi_n) = \mathrm{Out}^{GF}(\Pi_n) \times \mathfrak{S}_{n+3}$

(ii) • $\mathfrak{S}_{n+3} = Z_{\mathrm{Out}(\Pi_n)}\left(\mathrm{Out}^{GF}(\Pi_n)\right)$

$= Z^{loc}\left(\mathrm{Out}(\Pi_n)\right)$

• $\mathrm{Out}^{GF}(\Pi_n) = Z_{\mathrm{Out}(\Pi_n)}\left(Z^{loc}(\mathrm{Out}(\Pi_n))\right)$

(iii) $\mathrm{Out}(\Pi_n) = \underset{\sim}{GT} \times \mathfrak{S}_{n+3}$

[profinite or pro-ℓ] Grothendieck-Teichmüller gp

$\mathrm{Out}^{GF}(\Pi_n) \times \mathfrak{S}_{n+3}$

symm gp

$= Z_{\mathrm{Out}(\Pi_n)}\left(\mathrm{Out}^{GF}(\Pi_n)\right)$

$= Z^{loc}\left(\mathrm{Out}(\Pi_n)\right)$

$(\Pi_n) = Z_{\mathrm{Out}(\Pi_n)}\left(Z^{loc}\left(\mathrm{Out}(\Pi_n)\right)\right)$

$= \underset{\sim}{GT} \times \mathfrak{S}_{n+3}$

[profinite or pro-ℓ] Grothendieck-Teichmüller gp

By $\boxed{\text{Thm B}}$, we obtain an ext seq

$$1 \longrightarrow \mathrm{Out}^{GF}(\Pi_n) \longrightarrow \mathrm{Out}(\Pi_n) \longrightarrow \mathfrak{S}_{n+3} \longrightarrow 1$$

$\exists$ section arising from sch theory

Considering the action on the set of gen fib subgps of co-length $= 1$

Thus (i) follows from (ii)

We consider (ii)

$$\mathfrak{S}_{n+3} \leq Z_{\mathrm{Out}}(\mathrm{Out}^{GF}) \leq Z^{loc}(\mathrm{Out}) \leq \mathfrak{S}_{n+3}$$

Combinatorial anabelian geom

[cf. S. Mochizuki Comb Cusp §4]

by def + $\boxed{\text{Thm B}}$

[arith] Grothendieck Conj / NF

//

(iii) $\mathrm{Out}^{GF}(\Pi_n) = GT$

follows from

$\mathrm{Out}^{GF} = \mathrm{Out}^{FC}$ (cf. Combinatorial topics II Thm A)

$= GT$ (cf Harbater – Schneps)

//

$\boxed{\text{Thm C}}$ $n \geq 2$

sym gp

(i) $\mathrm{Out}(\Pi_n) = \mathrm{Out}^{GF}(\Pi_n) \times \mathfrak{S}_{n+3}$

(ii) • $\mathfrak{S}_{n+3} = Z_{\mathrm{Out}(\Pi_n)}\left(\mathrm{Out}^{GF}(\Pi_n)\right)$

$= Z^{loc}\left(\mathrm{Out}(\Pi_n)\right)$

• $\mathrm{Out}^{GF}(\Pi_n) = Z_{\mathrm{Out}(\Pi_n)}\left(Z^{loc}\left(\mathrm{Out}(\Pi_n)\right)\right)$

(iii) $\mathrm{Out}(\Pi_n) = \underset{\sim}{GT} \times \mathfrak{S}_{n+3}$

[profinite or pro-ℓ] Grothendieck-Teichmüller gp

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